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The Monodromy matrices of the XXZ model in the infinite volume limit

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Abstract. We consider the XXZ model in the infinite volume limit with spin-$\frac{1}{2}$ quantum space and higher spin auxiliary space. Using perturbation theory arguments, we relate the half infinite transfer matrices of this class of model to certain $U_q(\widehat{sl_2})$ intertwiners introduced by Nakayashiki. We construct the monodromy matrices, and show that the one with spin-1 auxiliary space gives rise to the L-operator.

1. Introduction

In this paper, we revisit the $U_q(\widehat{sl_2})$ symmetry of the XXZ Hamiltonian in the infinite volume limit. We show that the monodromy matrix, in the sense used in the quantum inverse scattering method, gives rise to the L-operator representing the level 0 action of $U'_q(\widehat{sl_2})$ on the physical space of states. In this construction, we take the auxiliary space for the monodromy matrix to be the spin-1 $U'_q(\widehat{sl_2})$ module $V^{(2)}_\zeta$, i.e. the three-dimensional evaluation module with spectral parameter ζ. The monodromy matrix acts on the quantum space $\mathcal{F}$, which is the $N \to \infty$ limit of the N-fold tensor product of spin-$\frac{1}{2}$ $U'_q(\widehat{sl_2})$ modules $V^{(1)}$.

If we take the spin-$\frac{1}{2}$ module $V^{(1)}_\zeta$ as the auxiliary space, and consider the trace of the monodromy matrix acting on a finite, say N-fold, tensor product, we obtain the transfer matrix $T^{(1)}_N(\zeta)$ for the six-vertex model of size N. $T^{(1)}_N(\zeta)$ form a commuting family of operators which contains the XXZ Hamiltonian. In general, for any positive integer m, one can define a family of operators $T^{(m)}_N(\zeta)$ which commute with $T^{(1)}_N(\zeta)$ and among themselves, by choosing the spin $\frac{m}{2}$ auxiliary space $V^{(m)}_\zeta$. However, these are not new operators because the fusion relation expresses them as polynomials in $T^{(1)}_N(\zeta)$ with suitably shifted parameters. On the other hand, the components of the monodromy matrix, other than its trace, do not commute among themselves and, in fact, if $m = 1$, they obey the commutation relation of the L-operator.

The situation is different in the infinite volume limit. This is because one must choose appropriate boundary conditions in order to have a well-defined limit of the monodromy matrix. For example, suppose we take the spin-$\frac{1}{2}$ auxiliary space. We restrict our discussion to the massive case, i.e. $-1 < q < 0$. The dominant Boltzmann weight in this regime is the c weight (in the usual terminology of the six-vertex model). Let us normalize it to be

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1. The weights a and b are then small. To have a non-zero contribution, we must take all but a finite number of constituents of the monodromy matrix to be the c weight. Therefore, the choice of the boundary condition in the quantum space uniquely fixes the choice in the auxiliary space. In other words, the total spin of the auxiliary space at the boundaries effectively changes to 0.

We denote the monodromy matrix in this sense, acting on the infinite tensor product, by $T^{(1)}(\zeta)$, and call it the monodromy matrix in the infinite volume limit. The XXZ Hamiltonian is obtained from the first derivative of $T^{(1)}(\zeta)$ at $\zeta = 1$. In short, the transfer matrix in the infinite volume limit is nothing but the monodromy matrix with the spin-$\frac{1}{2}$ auxiliary space.

In general, the effective total spin of the monodromy matrix at the boundaries is equal to $(n-1)/2$ if we take the auxiliary space to be $V_\zeta^{(n)}$. We denote this operator by $T^{(n)}(\zeta) = (T_{l,l'}^{(n)}(\zeta))_{l,l'=0,\ldots,n-1}$. The main result of this paper is to show that the monodromy matrix $T^{(2)}(\zeta)$ can be interpreted as the L-operator.

The physical space of states for the XXZ model consists of the vacuum vectors and the multi-particle states. Particles have spin-$\frac{1}{2}$. Namely, they transform according to the two-dimensional evaluation representation of $U_q'(\widehat{sl_2})$ and constitute a space of states which is isomorphic to the tensor product of $V_{\xi_i}^{(1)}$ $(i = 1, \ldots, m)$. This is called the particle picture of the space of states. The action of the transfer matrix $T^{(1)}(\zeta)$ is diagonalized in the particle picture. It is 2^m-fold degenerate on each m-particle space $V_{\xi_1}^{(1)} \otimes \cdots \otimes V_{\xi_m}^{(1)}$ with a given set of spectral parameters $\{\xi_1, \ldots, \xi_m\}$. In order to resolve this degeneracy, we need the $U_q'(\widehat{sl_2})$ symmetry. We calculate the action of the monodromy matrix $T^{(n)}(\zeta)$ on $V_{\xi_1}^{(1)} \otimes \cdots \otimes V_{\xi_m}^{(1)}$ explicitly. The result is essentially equal to the action of the monodromy matrix of size m with spin $(n-1)/2$ auxiliary space.

Now, let us come to the representation theoretical content of the story. The two key observations in the series of works [1–6] on the XXZ model are that the half infinite tensor product of $V^{(1)}$ can be identified with the level 1 integrable highest weight representations $\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1)$ of $U_q(\widehat{sl_2})$, and that the half infinite transfer matrix acting on it is identified with an intertwiner called the type-I vertex operator. The space of states is given as $\mathcal{F} = \mathcal{H} \otimes \mathcal{H}^*$. This is called the local picture of the space of states. In [7], a new class of intertwiners is introduced, which generalizes the type-I vertex operator. For each integer $n \geqslant 0$, we consider Nakayashiki's intertwiner

$$\Phi^{(n)}(\zeta) : V_\zeta^{(n)} \otimes V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V_\zeta^{(n+1)}.$$

We show (up to a few orders in q) that the infinite volume limit of the half transfer matrix with auxiliary space $V_\zeta^{(n+1)}$ is identified with $\Phi^{(n)}(\zeta)$ for $n = 0, 1$, and conjecture that this statement is valid to all orders for all n. From this follows the representation theoretical definition of the monodromy matrix $T^{(n+1)}(\zeta)$ (see (2.10)). We then compute the action of $T^{(n)}(\zeta)$ on the space of states in the particle picture, and thereby derive the commutation relations of $T^{(n)}(\zeta)$. Finally, we derive the fusion relation which expresses $T^{(n)}(\zeta)$ in terms of $T^{(2)}(\zeta)$ and $T^{(1)}(\zeta)$ with suitably shifted parameters.

2. Half transfer matrices

Consider the six-vertex model specified by the following Boltzmann weights:

$$\tilde{a} = \tilde{R}_{0,0}^{0,0} = \tilde{R}_{1,1}^{1,1} = \frac{1-q^2\zeta^2}{\zeta(1-q^2)},$$

$$\tilde{b} = \tilde{R}^{0,1}_{0,1} = \tilde{R}^{1,0}_{1,0} = \frac{q(1-\zeta^2)}{\zeta(1-q^2)},$$
$$\tilde{c} = \tilde{R}^{0,1}_{1,0} = \tilde{R}^{1,0}_{0,1} = 1.$$

We restrict our consideration to the parameter region $-1 < q < 0$ and $1 < \zeta < -q^{-1}$. The R-matrix $\tilde{R} = (\tilde{R}^{k'_1,k'_2}_{k_1,k_2})$ acts on the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$. Following the terminology of the quantum inverse scattering method, we call the first component of the tensor product the auxiliary space, and the second component the quantum space. The monodromy matrix $T_N{}^l_{l'}$ ($l, l' = 0, 1$) is an operator acting on the N-fold tensor product of the quantum space $\mathbb{C}^2 = \mathbb{C}u^{(1)}_0 \oplus \mathbb{C}u^{(1)}_1$:

$$T_N{}^l_{l'}(u^{(1)}_{k_N} \otimes \cdots \otimes u^{(1)}_{k_1}) = \sum_{\substack{k'_1,\ldots,k'_N \\ l_1,\ldots,l_{N-1}}} \tilde{R}^{l_1,k_1}_{l',k'_1} \ldots \tilde{R}^{l,k_N}_{l_{N-1},k'_N} u^{(1)}_{k'_N} \otimes \cdots \otimes u^{(1)}_{k'_1}.$$

In this section, we use a small q expansion to compare the $N \to \infty$ limit of the monodromy matrix of size N with the level 1 intertwiners. We take the limit keeping the spin variable at one end fixed and changing the one at the other end according to the choice of boundary condition. This limit gives the action of the half transfer matrix in the infinite volume limit.

2.1. The half infinite tensor product

We make the identification of the half infinite tensor product $\lim_{N\to\infty}(\mathbb{C}^2)^{\otimes N}$ with the level 1 $U_q(\widehat{sl_2})$ module $\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1)$ in the way discussed in [1, 2, 5]. For $i = 0, 1$, consider the set of paths $\mathcal{P}^{(i)}$ consisting of sequences of 0, 1, denoted by $|p\rangle_{(i)} = \{p(j)\}_{j\geqslant 1}$, which satisfy the boundary condition $p(j) = (1-(-1)^{i+j})/2$ for sufficiently large j. Consider the vector space $\mathcal{H}^{(i)}$ spanned by the formal expressions $\sum_{p\in\mathcal{P}^{(i)}} c(p)|p\rangle_{(i)}$, where $c(p)$ is a formal power series in q. The statement (though not a mathematical theorem) is that there is an embedding κ of the irreducible highest weight module $V(\Lambda_i)$ into $\mathcal{H}^{(i)}$ such that the action of $U_q(\widehat{sl_2})$ on $V(\Lambda_i)$ is induced by the formal action of $U'_q(\widehat{sl_2})$ on the half infinite tensor product given by the coproduct (3.1).

For example, the path expansion of the highest weight vector $|\Lambda_0\rangle$ reads

$$\kappa(|\Lambda_0\rangle) = \cdots - q\sum \cdots (2) \cdots + q^2\left(\sum \cdots (2) \cdots (2) \cdots + 2\sum \cdots (4) \cdots\right)$$
$$+ q^3\left(\sum \cdots (2) + 2\sum_{k\geqslant 1} \cdots (2)(k) - \sum \cdots (1)(2)(1) \cdots\right) + O(q^4). \quad (2.1)$$

The notation is as follows. We call an alternating sequence of 0, 1 of maximal length a domain. We decompose a path into domains. In the above formula, we denote by (k) a domain of length k. The symbol $\cdots$, when it is placed at the leftmost end, means an infinite domain. Otherwise $\cdots$ means a domain of undetermined length. The length of such a domain can be any strictly positive integer. In addition, the length can be 0 if $\cdots$ is placed at the rightmost end.

The path expansion of the vectors in $V(\Lambda_1)$ are obtained by the $0 \leftrightarrow 1$ symmetry. In particular, the expansion of $\kappa(|\Lambda_1\rangle)$ in the above notation is exactly the same as (2.1).

2.2. The perturbative action of the half transfer matrix

For each N, we set

$$\mathcal{P}^{(i)}_N = \{|p\rangle_{(i)};\ p(j) = (1-(-1)^{i+j})/2 \text{ if } j > N\}.$$

We denote by ρ_N the projection of $\mathcal{H}^{(i)}$ to the vector space spanned by $\mathcal{P}_N^{(i)}$. We define an operator $\tilde{\Phi}_{N,k}(\zeta)$ acting from $\rho_N(\mathcal{H}^{(i)})$ to $\rho_N(\mathcal{H}^{(1-i)})$ by

$$\tilde{\Phi}_{N,k}(\zeta)|p\rangle_{(i)} = \sum_{|p'\rangle_{(1-i)}\in\mathcal{P}_N^{(1-i)}} \sum_{k_1,\ldots,k_{N-1}} \prod_{1\leqslant j\leqslant N} \tilde{R}^{k_j,p(j)}_{k_{j-1},p'(j)}(\zeta,q)|p'\rangle_{(1-i)} \tag{2.2}$$

where

$$k=0,1 \qquad k_0=k \qquad k_N=\tfrac{1}{2}(1-(-1)^{i+N+1}).$$

Since $\tilde{c}=1$, the matrix element of the half transfer matrix between $|p\rangle_{(i)}$ and $|p'\rangle_{(1-i)}$ is uniquely determined by the formula (2.2) by taking a sufficiently large N. If we take $|p'\rangle_{(i)}$ instead of $|p'\rangle_{(1-i)}$ in (2.2), the matrix element vanishes in the limit $N\to\infty$ because $|a|,|b|<1$. Note also that for given $|p\rangle_{(i)}\in\mathcal{P}^{(i)}$ and $|p'\rangle\in\mathcal{P}^{(1-i)}$, the k_j $(1\leqslant j\leqslant N-1)$ are uniquely determined. Only one term in the second sum of (2.2) is non-vanishing.

Now, we apply formula (2.2) to a vector which belongs to $\mathcal{H}^{(i)}$. It is an infinite linear combination of $|p\rangle$, and the coefficient of $|p'\rangle$ in the right-hand side must be summed up with respect to these $|p\rangle$. This sum diverges in the limit $N\to\infty$. To have a finite sum we need to renormalize the Boltzmann weights. We use the normalization (3.5) of the R-matrix such that the partition function is 1 (see [5]). The expansion of the weights a, b, c reads as

$$\begin{aligned} a &= \zeta^{-1}+q^2(\zeta^{-3}-\zeta)+O(q^4),\\ b &= q(\zeta^{-1}-\zeta)+q^3(\zeta^{-3}-\zeta^{-1})+O(q^4),\\ c &= 1+q^2(\zeta^{-2}-1)+O(q^4). \end{aligned} \tag{2.3}$$

We define $\Phi_{N,k}(\zeta)$ as in (2.2) but with $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ replaced by a, b, c.

Let $\Phi_k(\zeta)$ be the type-I vertex operator (see (3.6) and (3.7)). We conjecture

$$\lim_{N\to\infty}\frac{1}{\mu(\zeta,q)}\Phi_{N,k}(\zeta)\circ\rho_N\circ\kappa = \kappa\circ\Phi_k(\zeta) \tag{2.4}$$

where $\mu(\zeta,q)$ is a series in q with Laurent polynomials in ζ as coefficients.

Let us check (2.4) on the vector $|\Lambda_0\rangle$:

$$\lim_{N\to\infty}\frac{1}{\mu(\zeta,q)}\Phi_{N,k}(\zeta)\circ\rho_N\circ\kappa(|\Lambda_0\rangle) = \kappa\circ\Phi_k(\zeta)(|\Lambda_0\rangle). \tag{2.5}$$

We find

$$\mu(\zeta,q)=1+q^2(\zeta^{-2}-1)+O(q^4).$$

First, we will check the coefficients of $\cdots$, $\cdots(2)$ and $\cdots(2)(m)$ $(m\geqslant 1)$ in (2.5) with $k=1$. The coefficients of these terms in $\Phi_{N,1}(\zeta)\circ\rho_N\circ\kappa(|\Lambda_0\rangle)$ are, modulo $O(q^4)$,

$c^N-(N-1)qabc^{N-2}$,

$(-q+2q^3)a^2c^{N-2}+abc^{N-2}+(N-3)q^2a^3bc^{N-4}-(N-3)qa^2b^2c^{N-4}$,

$(-q+2q^3)a^2c^{N-2}+(1+2q^2)abc^{N-2}-qb^2+(N-4)q^2a^3bc^{N-4}-(N-4)qa^2b^2c^{N-4}$,

respectively. Using (2.3) we obtain

$$\begin{aligned} \lim_{N\to\infty}\Phi_{N,1}(\zeta)\circ\rho_N\circ\kappa(|\Lambda_0\rangle) &= (1+q^2(\zeta^{-2}-1)+O(q^4))\\ &\quad\times\Bigg(\cdots+(-q+q^3)\cdots(2)+(-q+2q^3)\sum_{m\geqslant1}\cdots(2)(m)+\cdots\\ &\quad+\zeta^2q^3\cdots(2)+\cdots\Bigg). \end{aligned} \tag{2.6}$$

Note that

$$\Phi(\zeta)|\Lambda_0\rangle = |\Lambda_1\rangle \otimes u_1^{(1)} - q\zeta f_1|\Lambda_1\rangle \otimes u_0^{(1)} + \frac{q^4\zeta^2}{1+q^2} f_0 f_1|\Lambda_1\rangle \otimes u_1^{(1)} + \cdots. \tag{2.7}$$

Following the method in [1], we obtain

$$\kappa(f_0 f_1|\Lambda_1\rangle) = (q^{-1} + \cdots)\cdots(2) + \cdots.$$

Therefore, the result (2.6) is consistent with the conjecture.

Next we check the case $k = 0$ in (2.4) for the coefficient of $\cdots(1)$. It is, modulo $O(q^4)$,

$$bc^{N-1} + (-q + q^3)ac^{N-1} + (N-2)q^2a^2bc^{N-3} - (N-2)qab^2c^{N-3}.$$

Using (2.3) we obtain

$$\lim_{N\to\infty} \Phi_{N,0}(\zeta) \circ \rho_N \circ \kappa(|\Lambda_0\rangle) = (1 + q^2(\zeta^{-2} - 1) + O(q^4))(-q(1-q^2)\zeta \cdots (1) + \cdots). \tag{2.8}$$

On the other hand, we have

$$\kappa(f_1|\Lambda_1\rangle) = (1-q^2)\cdots(1) + \cdots,$$

in (2.7). Therefore, the result (2.8) is consistent with the conjecture (2.4).

2.3. The case with spin-1 auxiliary space

Let us calculate similar quantities for the spin-1 auxiliary space. We take the weight vectors $u_j^{(2)}$ $(j = 0, 1, 2)$ of the three-dimensional $U_q'(\widehat{sl_2})$ module $V_\zeta^{(2)}$ as in (3.2). This is the choice such that the matrix elements of the spin $(1, \frac{1}{2})$ R-matrix are

$$\tilde{A} = \tilde{R}_{0,0}^{0,0} = \tilde{R}_{2,1}^{2,1} = \frac{1}{\sqrt{1+q^2}} \frac{1-q^3\zeta^2}{\zeta(1-q^2)},$$
$$\tilde{B} = \tilde{R}_{0,1}^{0,1} = \tilde{R}_{2,0}^{2,0} = \frac{1}{\sqrt{1+q^2}} \frac{q(q-\zeta^2)}{\zeta(1-q^2)},$$
$$\tilde{C} = \tilde{R}_{1,0}^{1,0} = \tilde{R}_{1,1}^{1,1} = \frac{1}{\sqrt{1+q^2}} \frac{q(1-q\zeta^2)}{\zeta(1-q^2)},$$
$$\tilde{D} = \tilde{R}_{1,1}^{2,0} = \tilde{R}_{1,0}^{0,1} = \tilde{R}_{0,1}^{1,0} = \tilde{R}_{2,0}^{1,1} = 1.$$

We set

$$\tilde{\Phi}_{N,l,k}^{(1)}(\zeta)|p\rangle_{(i)} = \sum_{|p'\rangle_{(1-i)} \in \mathcal{P}_N^{(1-i)}} \sum_{k_1,\ldots,k_{N-1}} \prod_{1\leqslant j\leqslant N} \tilde{R}_{k_{j-1},p'(j)}^{k_j,p(j)}(\zeta, q)|p'\rangle_{(1-i)},$$

where

$$l = 0, 1, \qquad k = 0, 1, 2, \qquad k_0 = k, \qquad k_N = l + \tfrac{1}{2}(1 - (-1)^{i+N+1}).$$

Note that $\tilde{D}$ dominates the other weights. Therefore, if we choose $|p\rangle_{(i)}$ in $\mathcal{P}^{(i)}$, we must choose $|p'\rangle_{(1-i)}$ in $\mathcal{P}^{(1-i)}$. This is the same as in the spin-$\frac{1}{2}$ case. A new phenomenon is that there are two choices of the value of k_N corresponding to $l = 0, 1$. In order to obtain finite results we define $\Phi_{N,l,k}^{(1)}(\zeta)$ by using the normalization (3.5) for the R-matrix. The overall normalization factor (corresponding to the $\mu(\zeta, q)$ in the case of the spin-$\frac{1}{2}$ auxiliary space) now depends upon whether N is odd or even. The conjecture is

$$\lim_{N\to\infty} \frac{1}{\nu_N(\zeta, q)} \Phi^{(1)}_{N,l,k}(\zeta) \circ \rho_N \circ \kappa = \kappa \circ \Phi^{(1)}_{l,k}(\zeta),$$

$$\nu_N(\zeta, q) = 1 - \left(\frac{N}{2} - \left[\frac{N}{2}\right]\right) q^2 + q^3\zeta^{-2} + O(q^4),$$

where we use Nakayashiki's operator $\Phi^{(1)}(\zeta)$ (see (3.8)) in the right-hand side. In the above formula the symbol $[\frac{N}{2}]$ means the integer part of $N/2$.

The following are the supporting calculations. Let us check the case $l = 1$, $k = 2$ on the vector $|\Lambda_0\rangle$. We have

$$\Phi^{(1)}_{1,2}(\zeta)|\Lambda_0\rangle = |\Lambda_1\rangle + \cdots.$$

The coefficients of $\cdots$, $\cdots(2)$, $\cdots(2)(m)$ $(m \geqslant 2,\ \text{even})$ and $\cdots(2)(m)$ $(m \geqslant 1,\ \text{odd})$ in $\Phi^{(1)}_{N,1,2}(\zeta) \circ \rho_N \circ \kappa(|\Lambda_0\rangle)$ are, modulo $O(q^4)$,

$$D^N - \left[\frac{N}{2}\right] qABD^{N-2} - \left[\frac{N-1}{2}\right] qC^2D^{N-2} \equiv \nu_N(\zeta, q),$$

$$ABD^{N-2} - qACD^{N-2} - \left(\left[\frac{N}{2}\right] - 1\right) qA^2B^2D^{N-4} \equiv (-q + q^3)\nu_N(\zeta, q),$$

$$-qBCD^{N-2} + ABD^{N-2} - qACD^{N-2} - \left(\left[\frac{N}{2}\right] - 1\right) qA^2B^2D^{N-4}$$
$$\equiv (-q + 2q^3)\nu_N(\zeta, q),$$

$$C^2D^{N-2} - qBCD^{N-2} + (-q + 2q^3)D^N - qACD^{N-2} + \left[\frac{N}{2}\right] q^2ABD^{N-2}$$
$$\equiv (-q + 2q^3)\nu_N(\zeta, q),$$

respectively. These are consistent with the conjecture.

Let us check the case $l = 0$, $k = 1$ on the vector $|\Lambda_0\rangle$. We have

$$\Phi^{(1)}_{0,1}(\zeta)|\Lambda_0\rangle = \left(1 - \frac{q^2}{2}\right)|\Lambda_1\rangle + \cdots.$$

The perturbative calculation gives the coefficient of $\cdots$ in $\Phi^{(1)}_{N,0,1}(\zeta) \circ \rho_N \circ \kappa(|\Lambda_0\rangle)$ to be, modulo $O(q^4)$,

$$D^N - \left[\frac{N}{2}\right] qC^2D^{N-2} - \left[\frac{N-1}{2}\right] qABD^{N-2} \equiv \left(1 - \frac{q^2}{2}\right)\nu_N(\zeta, q).$$

This is consistent with the conjecture.

Based on the perturbative checks in this and the previous subsection, we conjecture that the half transfer matrix with spin $(n+1)/2$ auxiliary space is given by Nakayashiki's operator $\Phi^{(n)}(\zeta)$. More specifically, we define the half transfer matrix with spin $(n+1)/2$ by

$$\Phi^{(n)}_{Nl,k}(\zeta)|p\rangle_{(i)} = \sum_{|p'\rangle_{(1-i)}\in\mathcal{P}_N^{(1-i)}} \sum_{k_1,\ldots,k_{N-1}} \prod_{N\leqslant j\leqslant 1} R^{(n+1,1)}(\zeta)^{k_j,p(j)}_{k_{j-1},p'(j)}|p'\rangle_{(1-i)}, \tag{2.9}$$

where $0 \leqslant l \leqslant n$, $0 \leqslant k \leqslant n+1$, $k_0 = k$, $k_N = l+\frac{1}{2}(1-(-1)^{N+i+1})$, and where $R^{(n+1,1)}(\zeta)$ is given in (3.5). Our conjecture is that $\lim_{N\to\infty} \Phi^{(n)}_{Nl,k}(\zeta)\circ\rho_N\circ\kappa$ is proportional to $\kappa\circ\Phi^{(n)}_{l,k}(\zeta)$, where $\Phi^{(n)}_{l,k}(\zeta)$ is given by (3.8) and (3.10).

2.4. *The monodromy matrix in the infinite volume limit*

Consider the monodromy matrix of size $2N$,

$$T^{(n+1)}_{2Nl,l'}(\zeta)^{p(N),\ldots,p(1-N)}_{p'(N),\ldots,p'(1-N)} = \sum_{k_{N-1},\ldots,k_{1-N}} \prod_{N\leqslant j\leqslant 1-N} R^{(n+1,1)}(\zeta)^{k_j,p(j)}_{k_{j-1},p'(j)},$$

where $k_N = l + \frac{1}{2}(1-(-1)^{N+i+1})$, $k_{-N} = l' + \frac{1}{2}(1-(-1)^{N+i+1})$, $0 \leqslant l,\ l' \leqslant n$ and $p(j),\ p'(j) = 0, 1$. Using (2.9), and the symmetry $R^{(n+1,1)}(\zeta)^{a,b}_{c,d} = R^{(n+1,1)}(\zeta)^{n+1-c,1-d}_{n+1-a,1-b}$, we can rewrite this as

$$T^{(n+1)}_{2Nl,l'}(\zeta)^{p(N),\ldots,p(1-N)}_{p'(N),\ldots,p'(1-N)} = \sum_{k=0}^{n+1} \Phi^{(n)}_{Nl,k}(\zeta)^{p(1),\ldots,p(N)}_{p'(1),\ldots,p'(N)} \otimes \Phi^{(n)}_{Nn-l',n+1-k}(\zeta)^{1-p'(0),\ldots,1-p'(1-N)}_{1-p(0),\ldots,1-p(1-N)}.$$

This observation, together with the conjectural form of $\lim_{N\to\infty}\Phi^{(n)}_{Nl,k}(\zeta)$, motivates us to define the monodromy matrix in the infinite volume limit, $T^{(n+1)}_{l,l'}(\zeta) \in \mathrm{End}(\mathcal{H}\otimes\mathcal{H}^*)$, as

$$T^{(n+1)}_{l,l'}(\zeta) = g^{(n)} \sum_{k=0}^{n+1} \Phi^{(n)}_{l,k}(\zeta) \otimes \Phi^{(n)t}_{n-l',n+1-k}(\zeta), \tag{2.10}$$

where $g^{(n)}$ is a constant which appears in the next section. This is a generalization of (7.8) (the case $n = 0$) in [5].

3. Level-1 intertwiners

In this section, we discuss level-1 intertwiners of the algebra $U' = U'_q(\widehat{sl_2})$ generated by e_i, f_i, t_i $(i = 0, 1)$. Unless otherwise stated, all notational conventions are those of [5]. We choose the coproduct Δ to be

$$\Delta(t_i) = t_i \otimes t_i, \qquad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \qquad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i. \tag{3.1}$$

We need two types of U' modules for our analysis; level-1 highest-weight modules, and level-zero evaluation modules. Level-1 highest-weight modules $V(\Lambda_j)$ $(j = 0, 1)$ are generated by the highest-weight vector v_{Λ_j} which obeys $e_i v_{\Lambda_j} = 0$, $t_i v_{\Lambda_j} = q^{\delta_{i,j}} v_{\Lambda_j}$ and $f_i^{\delta_{i,j}+1} v_{\Lambda_j} = 0$.

3.1. *Evaluation modules*

We use a principally specialized evaluation module $V^{(n)}_\zeta$, with weight vectors $u^{(n)}_l$. The action of U' on $V^{(n)}_\zeta$ is given by

$$\begin{aligned} f_1 u^{(n)}_l &= \zeta^{-1} b^{(n)}_l u^{(n)}_{l+1}, \qquad & e_1 u^{(n)}_l &= \zeta b^{(n)}_{n-l} u^{(n)}_{l-1}, \qquad & t_1 u^{(n)}_l &= q^{n-2l} u^{(n)}_l, \\ f_0 u^{(n)}_l &= \zeta^{-1} b^{(n)}_{n-l} u^{(n)}_{l-1}, \qquad & e_0 u^{(n)}_l &= \zeta b^{(n)}_l u^{(n)}_{l+1}, \qquad & t_0 &= t_1^{-1}, \end{aligned} \tag{3.2}$$

where $b^{(n)}_l = q^{(-n+2l+1)/2}([l+1][n-l])^{1/2}$. In this basis we have an isomorphism

$$V_{-q^{-1}\zeta} \to V^{*a}_\zeta, \qquad u^{(n)}_l \mapsto u^{(n)*}_{n-l},$$

where $\langle u^{(n)*}_l, u^{(n)}_k\rangle = \delta_{lk}$.

We now consider the structure of tensor products of such modules. First we note that $V^{(1)}_{q\zeta} \otimes V^{(1)}_\zeta$ has a one-dimensional U' submodule

$$M \hookrightarrow V^{(1)}_{q\zeta} \otimes V^{(1)}_\zeta, \qquad \text{where } M = \mathbb{C}(u^{(1)}_0 \otimes u^{(1)}_1 - u^{(1)}_1 \otimes u^{(1)}_0).$$

Letting $N_j = V^{(1)}_{q^{\frac{n-1}{2}}\zeta} \otimes V^{(1)}_{q^{\frac{n-3}{2}}\zeta} \otimes \cdots \otimes M \otimes \cdots \otimes V^{(1)}_{q^{\frac{1-n}{2}}\zeta}$, where M is the corresponding submodule in the $(j, j+1)$ position, we have

$$V^{(n)}_\zeta \simeq V^{(1)}_{q^{\frac{n-1}{2}}\zeta} \otimes V^{(1)}_{q^{\frac{n-3}{2}}\zeta} \otimes \cdots \otimes V^{(1)}_{q^{\frac{1-n}{2}}\zeta} \Big/ \sum_{j=1}^{n-1} N_j.$$

The U' linear projection $\pi^{(n)} : V^{(1)}_{q^{\frac{n-1}{2}}\zeta} \otimes V^{(1)}_{q^{\frac{n-3}{2}}\zeta} \otimes \cdots \otimes V^{(1)}_{q^{\frac{1-n}{2}}\zeta} \longrightarrow V^{(n)}_\zeta$ is given by

$$\pi^{(n)}(u^{(1)}_{l_1} \otimes u^{(1)}_{l_2} \otimes \cdots \otimes u^{(1)}_{l_n}) = \gamma^{(n)}_{l_1+\cdots+l_n} u^{(n)}_{l_1+l_2+\cdots+l_n},$$

where $\gamma^{(n)}_l = \begin{bmatrix} n \\ l \end{bmatrix}^{-\frac{1}{2}}$.

On the other hand, $V^{(1)}_{q^{\frac{1-n}{2}}\zeta} \otimes V^{(1)}_{q^{\frac{3-n}{2}}\zeta} \otimes \cdots \otimes V^{(1)}_{q^{\frac{n-1}{2}}\zeta}$ has a U' sub-module $V^{(n)}_\zeta$. The U' linear embedding $\iota^{(n)} : V^{(n)}_\zeta \hookrightarrow V^{(1)}_{q^{\frac{1-n}{2}}\zeta} \otimes V^{(1)}_{q^{\frac{3-n}{2}}\zeta} \otimes \cdots \otimes V^{(1)}_{q^{\frac{n-1}{2}}\zeta}$ is given by

$$\iota^{(n)}(u^{(n)}_l) = \gamma^{(n)}_l \tilde{u}^{(n)}_l \qquad \text{where } \tilde{u}^{(n)}_l = \sum_{l_1+\cdots+l_n=l} u^{(1)}_{l_1} \otimes u^{(1)}_{l_2} \otimes \cdots \otimes u^{(1)}_{l_n}.$$

We construct certain R-matrices associated with our evaluation modules. The R-matrix $\bar{R}^{(n,m)}(\zeta_1/\zeta_2) : V^{(n)}_{\zeta_1} \otimes V^{(m)}_{\zeta_2} \longrightarrow V^{(n)}_{\zeta_1} \otimes V^{(m)}_{\zeta_2}$ is defined up to a normalization by the requirement that $\bar{R}^{(n,m)}(\zeta_1/\zeta_2)\Delta(x) = \Delta'(x)\bar{R}^{(n,m)}(\zeta_1/\zeta_2)$ on $V^{(n)}_{\zeta_1} \otimes V^{(m)}_{\zeta_2}$. Here, $x \in U'$, and $\Delta'(x)$ is defined in [5]. On $V^{(n)}_{\zeta_1} \otimes V^{(m)}_{\zeta_2} \otimes V^{(p)}_{\zeta_3}$, we have the Yang–Baxter equation

$$\bar{R}^{(m,p)}(\zeta_2/\zeta_3)\bar{R}^{(n,p)}(\zeta_1/\zeta_3)\bar{R}^{(n,m)}(\zeta_1/\zeta_2) = \bar{R}^{(n,m)}(\zeta_1/\zeta_2)\bar{R}^{(n,p)}(\zeta_1/\zeta_3)\bar{R}^{(m,p)}(\zeta_2/\zeta_3). \tag{3.3}$$

We define components by

$$\bar{R}^{(n,m)}(\zeta_1/\zeta_2)u^{(n)}_l \otimes u^{(m)}_k = \sum_{l',k'} u^{(n)}_{l'} \otimes u^{(m)}_{k'} \bar{R}^{(n,m)}(\zeta_1/\zeta_2)^{l,k}_{l',k'},$$

and fix the normalization by requiring that $\bar{R}^{(n,m)}(\zeta)^{0,0}_{0,0} = 1$. The simplest way to obtain $\bar{R}^{(n,1)}(\zeta)$ is through the fusion technique. This gives

$$\bar{R}^{(n,1)}(\zeta)^{l,k}_{l',k'} = \frac{\gamma^{(n)}_{l'}}{\gamma^{(n)}_l} \sum_{l'_1+\cdots+l'_n=l',k_1,\ldots,k_{n-1}} \bar{R}(\zeta q^{\frac{n-1}{2}})^{l_1,k_1}_{l'_1,k'} \bar{R}(\zeta q^{\frac{n-3}{2}})^{l_2,k_2}_{l'_2,k_1} \ldots \bar{R}(\zeta q^{\frac{1-n}{2}})^{l_n,k}_{l'_n,k_{n-1}}, \tag{3.4}$$

where $l_1 + \cdots + l_n = l$, and $\bar{R}(\zeta) = \bar{R}^{(1,1)}(\zeta)$. Explicitly, we have

$$\bar{R}^{(n,1)}(\zeta)^{l,0}_{l,0} = \frac{q^l(1 - q^{n+1-2l}\zeta^2)}{1 - q^{n+1}\zeta^2}, \qquad \bar{R}^{(n,1)}(\zeta)^{l,1}_{l+1,0} = ([n-l][l+1])^{\frac{1}{2}} q^{\frac{n-1}{2}} \frac{(1-q^2)\zeta}{1 - q^{n+1}\zeta^2}.$$

Other components are given by the symmetries $\bar{R}^{(n,1)}(\zeta)^{a,b}_{c,d} = \bar{R}^{(n,1)}(\zeta)^{n-a,1-b}_{n-c,1-d} = \bar{R}^{(n,1)}(\zeta)^{c,d}_{a,b}$. The other R-matrix to which we shall refer later on is $\bar{R}^{(1,n)}(\zeta)$. This is given by $\bar{R}^{(1,n)}(\zeta)^{a,b}_{c,d} = \bar{R}^{(n,1)}(\zeta)^{b,a}_{d,c}$. Finally, we shall use the normalized R-matrices

$$R^{(n,1)}(\zeta) = \bar{R}^{(n,1)}(\zeta)/\kappa^{(n)}(\zeta), \qquad R^{(1,n)}(\zeta) = \bar{R}^{(1,n)}(\zeta)/\kappa^{(n)}(\zeta), \tag{3.5}$$

where

$$\kappa^{(n)}(\zeta) = \zeta \frac{(q^{3+n}\zeta^2; q^4)_\infty (q^{1+n}\zeta^{-2}; q^4)_\infty}{(q^{3+n}\zeta^{-2}; q^4)_\infty (q^{1+n}\zeta^2; q^4)_\infty}.$$

With this factor, $R^{(n,1)}(\zeta)$ enjoys the properties of unitarity

$$\sum_{l',k'} R^{(n,1)}(\zeta)^{l',k'}_{l_1,k_1} R^{(n,1)}(\zeta^{-1})^{l_2,k_2}_{l',k'} = \delta_{l_1,l_2}\delta_{k_1,k_2},$$

and crossing symmetry $R^{(n,1)}(-q^{-1}\zeta)^{l',k'}_{l,k} = R^{(n,1)}(\zeta^{-1})^{n-l,k'}_{n-l',k}$.

3.2. Elementary intertwiners

First, we recall the definition of the elementary U' intertwiners

$$\begin{aligned}\Phi(\zeta) &: V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_\zeta^{(1)},\\ \Psi^*(\zeta) &: V_\zeta^{(1)} \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{1-i})\end{aligned} \tag{3.6}$$

of [5, 2]. Components are defined by

$$\begin{aligned}\Phi(\zeta)v &= \sum_{k=0}^{1} \Phi_k(\zeta)v \otimes u_k^{(1)},\\ \Psi^*(\zeta)(u_k^{(1)} \otimes v) &= \Psi_k^*(\zeta)v,\end{aligned} \tag{3.7}$$

where $v \in V(\Lambda_i)$, and $\Phi_k(\zeta)$ and $\Psi_k^*(\zeta)$ are both maps $V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes \mathbb{C}[\zeta, \zeta^{-1}]$. Note that $k = 0, 1$ here corresponds to $\epsilon = +, -$ in [5]. These intertwiners are unique up to a normalization, which is fixed by the requirements

$$\begin{aligned}\langle\Lambda_1|\Phi_1(\zeta)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\Phi_0(\zeta)|\Lambda_1\rangle = 1,\\ \langle\Lambda_1|\Psi_0^*(\zeta)|\Lambda_0\rangle = 1, \qquad \langle\Lambda_0|\Psi_1^*(\zeta)|\Lambda_1\rangle = 1.\end{aligned}$$

Here we use the bra–ket notation, identifying $v_{\Lambda_i} = |\Lambda_i\rangle$.

3.3. General intertwiners

The existence and uniqueness of U' intertwiners of the form

$$\Phi^{(n)}(\zeta) : V_\zeta^{(n)} \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_\zeta^{(n+1)}$$

was demonstrated in [7]. Our construction differs from that of [7] only in that we use the principal evaluation representation. Following [7], we construct $\Phi^{(n)}(\zeta)$ in terms of the intertwiner $O^{(n)} : V_{\xi_1}^{(1)} \otimes \cdots \otimes V_{\xi_n}^{(1)} \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_{\zeta_1}^{(1)} \otimes \cdots \otimes V_{\zeta_{n+1}}^{(1)}$ defined by

$$O^{(n)} = \frac{1}{f^{(n)}} \Phi(\zeta_1) \cdots \Phi(\zeta_{n+1}) \Psi^*(\xi_1) \cdots \Psi^*(\xi_n),$$

where

$$\begin{aligned}f^{(n)} = \prod_a (-q^3 \xi_a^2)^{(n+1-a)/2} \prod_b (-q^3 \zeta_b^2)^{(1-b)/2} \prod_{a<a'} \frac{((\xi_{a'}/\xi_a)^2; q^4)_\infty}{(q^2(\xi_{a'}/\xi_a)^2; q^4)_\infty}\\ \times \prod_{b<b'} \frac{(q^2(\zeta_{b'}/\zeta_b)^2; q^4)_\infty}{(q^4(\zeta_{b'}/\zeta_b)^2; q^4)_\infty} \prod_{a,b} \frac{(q^3(\xi_a/\zeta_b)^2; q^4)_\infty}{(q(\xi_a/\zeta_b)^2; q^4)_\infty}.\end{aligned}$$

Here we adopt the convention that $a \in \{1, \ldots, n\}$ and $b \in \{1, \ldots, n+1\}$. We define $\Phi^{(n)}(\zeta)$ by

$$\Phi^{(n)}(\zeta) = (1 \otimes \pi^{(n+1)}) O^{(n)} (\iota^{(n)} \otimes 1)|_{\mathcal{S}}, \tag{3.8}$$

where $\mathcal{S}$ is the one-dimensional submanifold in the parameter space of $\{\xi_a, \zeta_b\}$:

$$\mathcal{S} = \{\xi_a = q^{-\frac{n+1}{2}+a}\zeta; \zeta_b = q^{\frac{n+2}{2}-b}\zeta\}.$$

As we show in the appendix, each component of $O^{(n)}$ has poles at $\mathcal{S}$. However, the combination $(1 \otimes \pi^{(n+1)}) O^{(n)} (\iota^{(n)} \otimes 1)$ is free of poles on $\mathcal{S}$.

Let U_i' ($i = 0, 1$) be the U' subalgebra generated by e_i, f_i and t_i, and define

$$\begin{aligned}\pi_i^{(n+1)} &: V_{\zeta_1}^{(1)} \otimes \cdots \otimes V_{\zeta_{n+1}}^{(1)} \to V_\zeta^{(n+1)},\\ \iota_i^{(n)} &: V_\zeta^{(n)} \to V_{\xi_1}^{(1)} \otimes \cdots \otimes V_{\xi_n}^{(1)}\end{aligned}$$

to be the unique U_i' intertwiners normalized by

$$\pi_i^{(n+1)}(u_0^{(1)} \otimes \cdots \otimes u_0^{(1)}) = u_0^{(n+1)},$$
$$\iota_i^{(n)}(u_0^{(n)}) = u_0^{(1)} \otimes \cdots \otimes u_0^{(1)}.$$

In order to prove that $\Phi^{(n)}(\zeta)$ defined by (3.8) is a U' intertwiner, it is enough to show that

$$\Phi^{(n)}(\zeta) = (1 \otimes \pi_i^{(n+1)}) O^{(n)} (\iota_i^{(n)} \otimes 1)|_{\mathcal{S}}. \tag{3.9}$$

The restriction to $\mathcal{S}$ in (3.9) is regular, and so the right-hand side of (3.9) is a U_i' intertwiner. In the appendix, we show that if we divide the restriction to $\mathcal{S}$ into the two steps

Step 1: $\xi_a = q^{\frac{1}{2}} \zeta_{n+2-a} \qquad (1 \leqslant a \leqslant n),$

Step 2: $\zeta_b = q^{\frac{n+2}{2}-b} \zeta \qquad (1 \leqslant b \leqslant n+1),$

then each component of $O^{(n)}$ is regular. The coefficients of the components in the linear combinations (3.9) (for both $i = 0$ and $i = 1$) and (3.8) coincide on $\mathcal{S}$. $\Phi^{(n)}(\zeta)$ is therefore an intertwiner for both U_0' and U_1'. This is a simple proof that $\Phi^{(n)}(\zeta)$ is a U' intertwiner.

Define components by

$$\Phi^{(n)}(\zeta)(u_l^{(n)} \otimes v) = \sum_{k=0}^{n+1} (\Phi_{l,k}^{(n)}(\zeta) v \otimes u_k^{(n+1)}). \tag{3.10}$$

The normalization of (3.8) is such that

$$\langle \Lambda_1 | \Phi_{n,n+1}^{(n)}(\zeta) | \Lambda_0 \rangle = 1, \qquad \langle \Lambda_0 | \Phi_{0,0}^{(n)}(\zeta) | \Lambda_1 \rangle = 1.$$

The intertwiner $\Phi^{(n)}(\zeta)$ has the following properties, analogous to those of the elementary intertwiner $\Phi(\zeta)$ given in [5] (indeed we can identify $\Phi(\zeta) = \Phi^{(0)}(\zeta)$):

$$g^{(n)} \sum_{k=0}^{n+1} \Phi_{l_1,k}^{(n)}(-q^{-1}\zeta) \Phi_{n-l_2,n+1-k}^{(n)}(\zeta) = \delta_{l_1,l_2}, \tag{3.11}$$

$$\sum_{l',k'} R^{(1,n+1)}(\zeta/\xi)_{l_1,k}^{l',k'} \Phi_{l'}(\zeta) \Phi_{l_2,k'}^{(n)}(\xi) = \Phi_{l_2,k}^{(n)}(\xi) \Phi_{l_1}(\zeta), \tag{3.12}$$

$$\sum_{l',k'} \Psi_{k'}^*(\xi) \Phi_{l',k_1}^{(n)}(\zeta) R^{(n,1)}(\zeta/\xi)_{l',k'}^{l,k_2} = \Phi_{l,k_1}^{(n)}(\zeta) \Psi_{k_2}^*(\xi), \tag{3.13}$$

$$\xi^{-D} \Phi_{l,k}^{(n)}(\zeta) \xi^D = \Phi_{l,k}^{(n)}(\zeta/\xi), \tag{3.14}$$

where

$$g^{(n)} = \frac{(q^{2+2n}; q^4)_\infty}{(q^{4+2n}; q^4)_\infty}.$$

In (3.14), D is the principal grading, which acts on $V(\Lambda_j)$ as

$$D(f_{i_1} f_{i_2} \ldots f_{i_N} v_{\Lambda_j}) = N(f_{i_1} f_{i_2} \ldots f_{i_N} v_{\Lambda_j}).$$

Relation (3.14) is a simple consequence of the analogous property for $\Phi(\zeta)$ and $\Psi^*(\zeta)$ (see [5]). Properties (3.11)–(3.13) can be derived by slightly modifying the proof of theorem 5 in [7]. We give a proof only of (3.11).

We use the following intertwiner:

$$\begin{aligned} &\bar{R}^{(n)}(\zeta_1, \ldots, \zeta_n) : V_{\zeta_1}^{(1)} \otimes \cdots \otimes V_{\zeta_n}^{(1)} \to V_{\zeta_n}^{(1)} \otimes \cdots \otimes V_{\zeta_1}^{(1)}, \\ &\bar{R}^{(n)}(\zeta_1, \ldots, \zeta_n)(u_0^{(1)} \otimes \cdots \otimes u_0^{(1)}) = u_0^{(1)} \otimes \cdots \otimes u_0^{(1)}, \end{aligned} \tag{3.15}$$

and abbreviate $\bar{R}^{(n+1)}(\zeta_1,\ldots,\zeta_{n+1})$ to R_ζ, and $\bar{R}^{(n)}(\xi_1,\ldots,\xi_n)$ to R_ξ. We denote the restriction to $\{\zeta_b = q^{\frac{n+2}{2}-b}\zeta\}$ by $|_\zeta$, and $\{\xi_a = q^{-\frac{n+1}{2}+a}\zeta\}$ by $|_\xi$. We write the duality map as

$$C^{(n)}: V^{(n)}_{-q^{-1}\zeta}\otimes V^{(n)}_\zeta \to \mathbb{C},$$
$$C^{(n)}(u^{(n)}_l\otimes u^{(n)}_{n-l}) = 1.$$

Then we have

$$(C^{(1)})^{n+1}(1\otimes R_\zeta)|_\zeta = C^{(n+1)}(\pi^{(n+1)}\otimes\pi^{(n+1)}), \tag{3.16}$$
$$(C^{(1)})^{n}(\iota^{(n)}\otimes R_\xi\iota^{(n)})|_\xi = C^{(n)}. \tag{3.17}$$

The following properties of elementary intertwiners are also known [5]:

$$g^{(0)}C^{(1)}\Phi(-q^{-1}\zeta)\Phi(\zeta) = \mathrm{id}, \tag{3.18}$$
$$\lim_{\xi_1\to -q^{-1}\xi_2}(1-q^{-2}\xi_2^2/\xi_1^2)\Psi^*(\xi_1)\Psi^*(\xi_2) = g^{(0)}C^{(1)}. \tag{3.19}$$

The proof of (3.11) proceeds as follows: In the notation of the appendix, the essential part of the left-hand side of (3.11) is

$$\left(\overline{\Phi(\zeta_1')\ldots\Phi(\zeta_{n+1}')\Psi^*(\xi_1')\ldots\Psi^*(\xi_n')}\right)\times\left(\overline{\Phi(\zeta_1)\ldots\Phi(\zeta_{n+1})\Psi^*(\xi_1)\ldots\Psi^*(\xi_n)}\right). \tag{3.20}$$

To get (3.11) we compose this with $C^{(n+1)}(\pi^{(n+1)}\otimes\pi^{(n+1)})$ and $\iota^{(n)}\otimes\iota^{(n)}$, and then restrict it to $\mathcal{S}$ and $\{\xi_a' = q^{-\frac{n+1}{2}+a}\zeta';\ \zeta_b' = q^{\frac{n+2}{2}-b}\zeta'\}$, and finally to $\zeta' = -q^{-1}\zeta$. Let us denote this restriction by $|_{\mathrm{restrict}}$. Consider the product of operators

$$\begin{aligned}&\overline{\Phi(\zeta_1')\ldots\Phi(\zeta_{n+1}')\Psi^*(\xi_1')\ldots\Psi^*(\xi_n')\Phi(\zeta_1)\ldots\Phi(\zeta_{n+1})\Psi^*(\xi_1)\ldots\Psi^*(\xi_n)}\\ &\qquad = \overline{\Phi(\zeta_1')\ldots\Phi(\zeta_{n+1}')\Phi(\zeta_1)\ldots\Phi(\zeta_{n+1})\Psi^*(\xi_1')\ldots\Psi^*(\xi_n')\Psi^*(\xi_1)\ldots\Psi^*(\xi_n)}.\end{aligned} \tag{3.21}$$

The two expressions (3.20) and (3.21) are equal up to a factor which is regular when we restrict in the way explained above. Therefore, we can manipulate

$$\begin{aligned}&C^{(n+1)}(\pi^{(n+1)}\otimes\pi^{(n+1)})\\ &\qquad\times\overline{\Phi(\zeta_1')\ldots\Phi(\zeta_{n+1}')\Phi(\zeta_1)\ldots\Phi(\zeta_{n+1})\Psi^*(\xi_1')\ldots\Psi^*(\xi_n')\Psi^*(\xi_1)\ldots\Psi^*(\xi_n)}\\ &\qquad\times(\iota^{(n)}\otimes\iota^{(n)})|_{\mathrm{restrict}}\end{aligned}$$

instead of the expression containing (3.20). Using (3.15) and (3.16), we reduce this expression to

$$\begin{aligned}&(C^{(1)})^{n+1}\overline{\Phi(\zeta_1')\ldots\Phi(\zeta_{n+1}')\Phi(\zeta_{n+1})\ldots\Phi(\zeta_1)\Psi^*(\xi_1')\ldots\Psi^*(\xi_n')\Psi^*(\xi_n)\ldots\Psi^*(\xi_1)}\\ &\qquad\times(\iota^{(n)}\otimes R_\xi\iota^{(n)})|_{\mathrm{restrict}}.\end{aligned}$$

Removing the bar, using (3.18), (3.19) and (3.17), and calculating the restriction of the contraction terms, we arrive at (3.11).

4. The monodromy matrices

In section 2, the results of perturbation theory and other considerations led us to define the monodromy matrix as $T^{(n+1)}_{l,l'}(\zeta) = g^{(n)}\sum_{k=0}^{n+1}\Phi^{(n)}_{l,k}(\zeta)\otimes\Phi^{(n)t}_{n-l',n+1-k}(\zeta)$. In section 3, following Nakayashiki, we defined $\Phi^{(n)}(\zeta)$ and presented its properties (3.11)–(3.14). We now use these results in order to derive certain properties of $T^{(n+1)}(\zeta)$.

4.1. Action on $\mathcal{F}$

One can view $\mathcal{F} = \mathcal{H} \otimes \mathcal{H}^*$ as a linear map on $\mathcal{H}$ via the canonical identification $\mathcal{H}^* \otimes \mathcal{H} \simeq \mathrm{End}(\mathcal{H})$. Then the action of $T^{(n+1)}_{l,l'}(\zeta)$ on $f \in \mathrm{End}(\mathcal{H})$ is given by

$$T^{(n+1)}_{l,l'}(\zeta) f = g^{(n)} \sum_{k=0}^{n+1} \Phi^{(n)}_{l,k}(\zeta) \circ f \circ \Phi^{(n)}_{n-l',n+1-k}(\zeta).$$

As an element of $\mathrm{End}(\mathcal{H})$, the vacuum in the ith sector was identified in [5] as

$$|vac\rangle_{(i)} = \chi^{-\frac{1}{2}}(-q)^{D^{(i)}} P^{(i)},$$

where $P^{(i)}$ is the projector $\mathcal{H} \to V(\Lambda_i)$, and $\chi = 1/(q^2; q^4)_\infty$ is the principally specialized character of $V(\Lambda_i)$. The superscript on the grading D serves only to indicate on which space $V(\Lambda_i)$ it acts (we suppress the appearance of the projector from now on).

The action of $T^{(n+1)}_{l,l'}(\zeta)$ is given by

$$\begin{aligned} T^{(n+1)}_{l,l'}(\zeta)|vac\rangle_{(i)} &= \chi^{-\frac{1}{2}} g^{(n)} \sum_{k=0}^{n+1} \Phi^{(n)}_{l,k}(\zeta)(-q)^{D^{(i)}} \Phi^{(n)}_{n-l',n+1-k}(\zeta), \\ &= \chi^{-\frac{1}{2}}(-q)^{D^{(1-i)}} g^{(n)} \sum_{k=0}^{n+1} \Phi^{(n)}_{l,k}(-q^{-1}\zeta) \Phi^{(n)}_{n-l',n+1-k}(\zeta), \\ &= \delta_{l,l'} \chi^{-\frac{1}{2}}(-q)^{D^{(1-i)}} = \delta_{l,l'}|vac\rangle_{(1-i)}. \end{aligned}$$

Here we have used properties (3.14) and (3.11).

The Hamiltonian of the XXZ model is given by

$$H = \frac{(1-q^2)}{2q} \zeta \frac{\mathrm{d}}{\mathrm{d}\zeta} T^{(1)}(\zeta)|_{\zeta=1}.$$

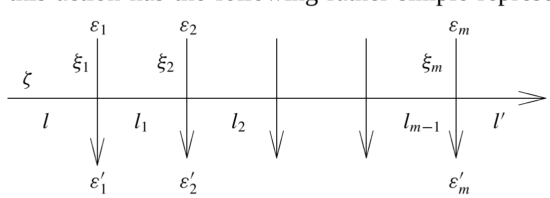
Excited states are given by

$$|\xi_1, \ldots, \xi_m\rangle_{\epsilon_1,\ldots,\epsilon_m;(i)} = (g^{(0)})^{-m/2} \chi^{-\frac{1}{2}} \Psi^*_{\epsilon_1}(\xi_1) \ldots \Psi^*_{\epsilon_m}(\xi_1)(-q)^{D^{(i)}}$$

with $|\xi_i| = 1$ (see [5]). Using the commutation relation (3.13), it is easy to show that the action of $T^{(n+1)}_{l,l'}(\zeta)$ on the 'm-particle state' $|\xi_1, \ldots, \xi_m\rangle_{\epsilon_1,\ldots,\epsilon_m;(i)}$ is given by

$$\begin{aligned} T^{(n+1)}_{l,l'}(\zeta)|\xi_1, \ldots, \xi_m\rangle_{\epsilon_1,\ldots,\epsilon_m;(i)} &= \sum_{\{\epsilon'_i, l_i\}} R^{(n,1)}(\zeta/\xi_1)^{l,\epsilon_1}_{l_1,\epsilon'_1} \\ &\times R^{(n,1)}(\zeta/\xi_2)^{l_1,\epsilon_2}_{l_2,\epsilon'_2} \ldots R^{(n,1)}(\zeta/\xi_m)^{l_{m-1},\epsilon_m}_{l',\epsilon'_m} |\xi_1, \ldots, \xi_m\rangle_{\epsilon'_1,\ldots,\epsilon'_m;(1-i)}. \end{aligned} \quad (4.1)$$

Here the sum is over $\epsilon'_1, \ldots, \epsilon'_m$ and $l_1, \ldots, l_{m-1}$. If we represent the R-matrix graphically as in [5], then this action has the following rather simple representation:



This picture is related to the space of particles and not to the coordinate lattice. For $n = 0$, we have $T^{(1)}(\zeta)|\xi_1, \ldots, \xi_m\rangle_{\epsilon_1,\ldots,\epsilon_m;(i)} = \tau(\zeta/\xi_1) \ldots \tau(\zeta/\xi_m)|\xi_1, \ldots, \xi_m\rangle_{\epsilon_1,\ldots,\epsilon_m;(1-i)}$ as in [5], where

$$\tau(\zeta) = \zeta^{-1} \frac{(q\zeta^2; q^4)_\infty (q^3\zeta^{-2}; q^4)_\infty}{(q\zeta^{-2}; q^4)_\infty (q^3\zeta^2; q^4)_\infty}.$$

4.2. Commutation relations

Making use of the explicit form of the action on $\mathcal{F}$ given by (4.1), and the Yang–Baxter equation (3.3), we can immediately write down the commutation relationships of $T^{(n)}_{l,l'}(\zeta)$.

$$\sum_{\bar{l}_1,\bar{l}_2} \bar{R}^{(n,m)}(\zeta_1/\zeta_2)^{l_1,l_2}_{\bar{l}_1,\bar{l}_2} T^{(m+1)}_{\bar{l}_2,l'_2}(\zeta_2) T^{(n+1)}_{\bar{l}_1,l'_1}(\zeta_1) = \sum_{\bar{l}_1,\bar{l}_2} T^{(n+1)}_{l_1,\bar{l}_1}(\zeta_1) T^{(m+1)}_{l_2,\bar{l}_2}(\zeta_2) \bar{R}^{(n,m)}(\zeta_1/\zeta_2)^{\bar{l}_1,\bar{l}_2}_{l'_1,l'_2},$$

$$[T^{(n)}(\zeta_1), T^{(1)}(\zeta_2)] = 0,$$

where $n, m \geqslant 1$. Thus $T^{(2)}_{l,l'}(\zeta)$ can be interpreted as the L-operator of the spin-$\frac{1}{2}$ XXZ model in the infinite volume limit.

4.3. Fusion

If we rewrite each of the $R^{(n,1)}(\zeta)$ that occur on the right-hand side of (4.1) using the fusion expression (3.4), then we obtain a fusion relation for $T^{(n+1)}(\zeta)$ in terms of $T^{(2)}(\zeta)$ and $T^{(1)}(\zeta)$.

Consider the operator F which counts the number of particles in the particle picture. Define

$$\bar{T}^{(n+1)}(\zeta) = \begin{cases} T^{(n+1)}(\zeta) T^{(1)}(\zeta)^{-1} & \text{if } n \equiv 0 \bmod 4 \\ T^{(n+1)}(\zeta) & \text{if } n \equiv 1 \bmod 4 \\ T^{(n+1)}(\zeta) T^{(1)}(\zeta) & \text{if } n \equiv 2 \bmod 4 \\ (-1)^F T^{(n+1)}(\zeta) & \text{if } n \equiv 3 \bmod 4. \end{cases} \tag{4.2}$$

We note that there is an equality $\bar{T}^{(n+1)}(\zeta) = T^{(n+1)}(\zeta) \prod_{a=0}^{n-2} T^{(1)}(q^{\frac{n-2}{2}-a}\zeta)$. Then, we have the following fusion relation:

$$\bar{T}^{(n+1)}_{l,l'}(\zeta) = \frac{\gamma^{(n)}_{l'}}{\gamma^{(n)}_{l}} \sum_{l'_1+\cdots+l'_n=l'} T^{(2)}_{l_1,l'_1}(\zeta q^{\frac{n-1}{2}}) T^{(2)}_{l_2,l'_2}(\zeta q^{\frac{n-3}{2}}) \ldots T^{(2)}_{l_n,l'_n}(\zeta q^{\frac{1-n}{2}}),$$

where $n \geqslant 1$. Here, the l_a are specified only by the requirement $l_1 + \cdots + l_n = l$; the formula is independent of the actual choice of l_a.

5. Discussion

In this paper, we have studied the $U_q(\widehat{sl_2})$ symmetry of the spin-$\frac{1}{2}$ XXZ model in the massive regime by making use of Nakayashiki's intertwiners. We have conjectured that in the infinite volume limit the half transfer matrix with spin-$\frac{1}{2}$ quantum space and spin-$(\frac{n+1}{2})$ auxiliary space is represented by the intertwiner

$$\Phi^{(n)}(\zeta) : V^{(n)}_\zeta \otimes V(\Lambda_i) \to V(\Lambda_{1-i} \otimes V^{(n+1)}_\zeta). \tag{5.1}$$

This implies, in particular, that the monodromy matrix with spin-1 auxiliary space enjoys the commutation relations of the L-operator.

In [7], Nakayashiki uses the operator $\Phi^{(n)}(\zeta)$ to diagonalize the spin-$\frac{1}{2}$ XXZ model with higher spin impurities. In the language of the six-vertex model, this is equivalent to inserting lines with higher spin. The difference between our approach and Nakayashiki's is that we consider the monodromy matrices which are parallel to the inserted lines, while Nakayashiki considers the transfer matrix which is perpendicular to them. In Nakayashiki's case the space $V^{(n)}_\zeta$ in (5.1) corresponds to the degeneracy of the vacuum states of the transfer matrix. In our case, we have found that the same space corresponds to the boundary conditions for the monodromy matrices.

We have derived the fusion relation for the monodromy matrices. It almost corresponds to the fusion construction of the space $V_\zeta^{(n)}$ in (5.1) out of the spaces with $n = 1$. However, we have found that the monodromy matrices contain the correction factor given in (4.2), which is diagonal in each irreducible m particle representation in the physical space of states.

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Appendix. Regularity of matrix elements

In this appendix, we prove a few statements concerning the regularity of the matrix elements of the product of type-I and type-II vertex operators. We start from the bosonization of vertex operators given on p 140 of [5]. Consider the product of vertex operators

$$O = \Phi_{k_1}(\zeta_1)\ldots\Phi_{k_m}(\zeta_m)\Psi^*_{l_1}(\xi_1)\ldots\Psi^*_{l_n}(\xi_n).$$

It contains integrals with respect to the variables $w_{\bar b}$ in $X^-(w_{\bar b})$, for $\bar b$ such that $k_{\bar b} = 0$, and $u_{\bar a}$ in $X^+(u_{\bar a})$, for $\bar a$ such that $l_{\bar a} = 0$.

After normal ordering, the integrand, which depends on the variables ζ_b, ξ_a, $w_{\bar b}$ and $u_{\bar a}$, consists of the following three parts:

(i) O_1, the contraction terms, pairwise in $\Phi_1\Phi_1$, $\Phi_1\Psi_1^*$, $\Psi_1^*\Phi_1$ or $\Psi_1^*\Psi_1^*$.

We have

$$O_1 = \prod_{b<b'}(-q^3\zeta_b^2)^{\frac12}\frac{(q^2\zeta_{b'}^2/\zeta_b^2;q^4)_\infty}{(q^4\zeta_{b'}^2/\zeta_b^2;q^4)_\infty}\prod_{a<a'}(-q^3\xi_b^2)^{\frac12}\frac{(\xi_{a'}^2/\xi_a^2;q^4)_\infty}{(q^2\xi_{a'}^2/\xi_a^2;q^4)_\infty}$$
$$\times\prod_{a,b}(-q^3\zeta_b^2)^{-\frac12}\frac{(q^3\xi_a^2/\zeta_b^2;q^4)_\infty}{(q\xi_a^2/\zeta_b^2;q^4)_\infty}.$$

This is a function of ζ_b, ξ_a. In the above setting, pairs of the form $\Psi_1^*\Phi_1$ do not appear because Φ_1 is always in the left of Ψ_1^*;

(ii) O_2, the contraction terms for the rest of the pairs.

We have

$$\prod_{\bar b<\bar b'}(w_{\bar b}-w_{\bar b'})(w_{\bar b}-q^2w_{\bar b'})\prod_{\bar a<\bar a'}(u_{\bar a}-u_{\bar a'})(u_{\bar a}-q^{-2}u_{\bar a'})$$
$$\times\prod_{\bar a,\bar b}\frac{1}{(w_{\bar b}-qu_{\bar a})(w_{\bar b}-q^{-1}u_{\bar a})}\prod_{\bar a,b}(u_{\bar a}-q^3\zeta_b)\prod_{a,\bar b}(w_{\bar b}-q^3\xi_a)$$
$$\times\prod_{a\leqslant\bar a}\frac{q}{u_{\bar a}-q^4\xi_a^2}\prod_{\bar a\leqslant a}\frac{1}{u_{\bar a}-q^2\xi_a^2}\prod_{b\leqslant\bar b}\frac{1}{q(w_{\bar b}-q^2\zeta_b^2)}\prod_{\bar b\leqslant b}\frac{1}{w_{\bar b}-q^4\zeta_b^2}.$$

(iii) O_3, the rest, which is a normal-ordered product of vertex operators with coefficients that are Laurent polynomials in ζ_b, ξ_a, $w_{\bar{b}}$ and $u_{\bar{a}}$.

The contours for the integrals are such that the $q^4\zeta_b^2$, $qu_{\bar{a}}$, $q^{-1}u_{\bar{a}}$ are inside, and the $q^2\zeta_b^2$ are outside of the contour for the $w_{\bar{b}}$ integration; the $q^2\xi_a^2$ are inside, and the $q^4\xi_a^2$, $qw_{\bar{b}}$, $q^{-1}w_{\bar{b}}$ are outside of the contour for the $u_{\bar{a}}$ integration.

Denote the quantity which is O with O_1 removed, by $\overline{O}$. Note that because of the commutation relation (A.3) of [5], $\Phi_{l_b}(\zeta_b)$ commutes with $\Psi^*_{k_a}(\xi_a)$ inside of $\overline{O}$.

Now let us examine the regularity of the matrix elements of $\overline{O}$. It is enough to consider the O_2 term in the integrand. The possible pinchings of the contours occur in the following four cases (which we list with the relevant factors in the integrand):

Case 1.

$$\frac{1}{(u_{\bar{a}} - q^4\xi_{a_1}^2)(u_{\bar{a}} - q^2\xi_{a_2}^2)} \qquad \text{at } \xi_{a_1}^2 = q^{-2}\xi_{a_2}^2, \text{ for } a_1 \leqslant \bar{a} \leqslant a_2,$$

Case 2.

$$\frac{1}{(w_{\bar{b}} - q^2\zeta_{b_1}^2)(w_{\bar{b}} - q^4\zeta_{b_2}^2)} \qquad \text{at } \zeta_{b_1}^2 = q^2\zeta_{b_2}^2, \text{ for } b_1 \leqslant \bar{b} \leqslant b_2,$$

Case 3.

$$\frac{w_{\bar{b}} - q^3\xi_a^2}{(w_{\bar{b}} - q^2\zeta_b^2)(w_{\bar{b}} - qu_{\bar{a}})(u_{\bar{a}} - q^2\xi_a^2)} \qquad \text{at } \zeta_{b_1}^2 = q\xi_a^2, \text{ for } b \leqslant \bar{b},\ \bar{a} \leqslant a,$$

Case 4.

$$\frac{u_{\bar{a}} - q^3\zeta_b^2}{(w_{\bar{b}} - q^2\zeta_b^2)(w_{\bar{b}} - q^{-1}u_{\bar{a}})(u_{\bar{a}} - q^2\xi_a^2)} \qquad \text{at } \zeta_{b_1}^2 = q^{-1}\xi_a^2, \text{ for } b \leqslant \bar{b},\ \bar{a} \leqslant a.$$

Cases 1 and 2 give rise to poles. They are, at most, simple because of the factor $\prod_{\bar{a}<\bar{a}'}(u_{\bar{a}} - u_{\bar{a}'})$ or $\prod_{\bar{b}<\bar{b}'}(w_{\bar{b}} - w_{\bar{b}'})$, respectively. Cases 3 and 4 are pole free because of the factor $w_{\bar{b}} - q^3\xi_a^2$ or $u_{\bar{a}} - q^3\zeta_b^2$, respectively. The final remark is that the restriction of $\overline{O}$ at $q\zeta_b^2 = \xi_{a_2}^2$ for any b is regular at $\xi_{a_1}^2 = q^{-2}\xi_{a_2}^2$, and the restriction at $q\zeta_{b_2}^2 = \xi_a^2$ for any a is regular at $\zeta_{b_1}^2 = q^2\zeta_{b_2}^2$. This is because of the following factors in the numerator:

$$u_{\bar{a}} - q^3\zeta_b^2 = (u_{\bar{a}} - q^2\xi_{a_2}^2) + q^2(\xi_{a_2}^2 - q\zeta_b^2)$$
$$w_{\bar{b}} - q^3\xi_a^2 = (w_{\bar{b}} - q^4\zeta_{b_2}^2) + q^3(q\zeta_{b_2}^2 - \xi_a^2).$$

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